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Site-disordered spin systems in the Gaussian variational approximation

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Abstract. We define a replica field theory describing finite dimensional site-disordered spin systems by introducing the notion of grand canonical disorder, where the number of spins in the system is random but quenched. A general analysis of this field theory is made using the Gaussian variational or Hartree–Fock method, and illustrated with several specific examples. Irrespective of the form of interaction between the spins this approximation predicts a spin-glass phase. We discuss the replica symmetric phase at length, explicitly identifying the correlator that diverges at the spin-glass transition. We also discuss the form of continuous replica symmetry breaking found just below the transition. Finally we show how an analysis of ferromagnetic ordering indicates a breakdown of the approximation.

1. Introduction

We consider the site-disordered models in finite dimensions with the following Hamiltonian:

$$H = -\frac{1}{2}\sum_{ij} J(r_i - r_j) S_i S_j \tag{1.1}$$

in which N Ising spins are fixed at random points, r_i, and are subject to a deterministic potential $J(r)$. For example, a positive $J(r)$ decaying with distance describes a dilute ferromagnetic system. Antiferromagnetic systems can also be treated, and because there is no lattice in this picture there will be no antiferromagnetic ordering. We also have in mind oscillatory RKKY-type interactions which cause frustration, as in the antiferromagnetic case. Therefore, the physics we address is related to the two types of ordering that can occur—ferromagnetic and spin glass—and to the relation between them. We shall discover that the approximation we use to solve the model is not completely reliable for the ferromagnetic system, nevertheless we present the general development for ferromagnetic interactions indicating where sign differences arise for the antiferromagnetic case. The Hamiltonian (1.1) could also be interpreted as an infinite range Sherrington–Kirkpatrick model [1] in which the bond strengths are correlated and have been chosen from a distribution very different from those we usually consider.

This Hamiltonian can describe experimental systems rather well. However, analytic studies of disordered spin systems are usually based on lattice models in which the bonds take random values, such as the Edwards–Anderson Hamiltonian [2]. Analytic work based on (1.1) has been hampered by the lack of a suitable field theoretic model (however, a

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lattice-based formulation has been proposed [3]). By considering a situation in which the number of spins in the system is random but quenched we are able to write a replica field theory for these site-disordered systems [4]. This field theory seems to be simpler than many of those coming from bond-disordered and diluted lattice models and should be accessible to many standard analytic techniques.

In this paper, to gain an overall picture of the model, we use the technique variously known as Gaussian variational, Hartree–Fock, random-phase approximation and other names. It is necessary to use a method more sophisticated than mean field theory because the field theory that describes the model is expressed in terms of the magnetic variables and to understand the spin-glass physics one has to look at composite operators. Mean field theory alone misses this spin-glass physics, whereas the Gaussian variational method indicates that a spin-glass transition is uniquitous. We find that using the Gaussian variational approximation the model can be solved in considerable detail, analytically in the high-temperature region, and with the help of numerics at low temperature. The general picture that emerges is in accord with physical expectations; however, there are some points that throw the reliability of the approximation into doubt. This is not surprising since, although the approximation is expected to be good in the case of Heisenburg spins with many components, here we apply it to one-component Ising spins. The infinite component spin case is discussed in a companion paper [5]. As we shall show, the problems with the approximation appear most clearly in two issues. First, we shall find that for a purely ferromagnetic interaction a spin-glass transition is predicted to occur at slightly higher temperature than the ferromagnetic transition and we present proof of the impossibility of such a situation. Secondly, the approximation yields results that are not as dependent on the range of interaction or dimension as one would expect. This should act as a caution given the rather widespread application of the method in the literature. For this reason and also because a thorough understanding of this simplest non-trivial approach is a prerequisite for further advances, we feel that this study is worthwhile despite the deficiencies of the approximation.

One of the interesting aspects of this work is that we are able to investigate the spin-glass transition in a finite-dimensional model. In the formalism we present, replicas enter the theory at an early stage and the spin-glass ordering is intimately connected with the symmetry in replica space. The role of replica symmetry breaking (RSB) is of great interest in spin glasses, where the correct description of three-dimensional materials is still controversial. Although RSB in the mean field theory for spin glasses is now well understood [6] and related to the proliferation of pure states of the system, in finite dimensions the picture is less clear. Alternative qualitative approaches based on droplets [7] view the spin-glass phase as a disguised ferromagnetic phase with only two underlying fundamental states. The role of RSB in systems which undergo a ferromagnetic phase transition is maybe even less clear. We might mention the random field Ising model [8], dilute ferromagnets [9] and the whole issue of renormalization flow in the presence of RSB [10]. Experimentally, the problem is reflected in the difficulty with the so-called re-entrant transitions [11]. Part of the theoretical difficulty in these cases lies in the lack of a simple model in which the ferromagnetic transition can be explicitly analysed in the presence of RSB. All these issues provide strong motivation to study the model (1.1).

Some of this work has already been briefly reported in [4], here we attempt a more pedagogic approach, discussing the basic issues more thoroughly and giving detailed examples. The paper is organised as follows, section 2 explains 'grand canonical disorder' constructs the field theory and discusses some of the measurement issues. In section 3 we perform the Gaussian variational analysis of the theory to derive the variational equations. These equations are discussed for a general potential in sections 4 and 5 which concern the replica symmetric (RS) and RSB cases respectively. In the remaining sections we illustrate our results with reference to some particularly simple interactions, namely Yukawa and RKKY-type. A discussion of problems with this approach along with some speculation, appears in the conclusion. Two appendices give details of the solution of the four-index correlator and of the stability analysis.

2. Grand canonical disorder

First, consider a model where the number of spins N is fixed: N spins S_i are placed randomly at positions r_i uniformly throughout a volume V. We refer to this type of disorder as canonical disorder, as the number of particles is the same for each realization of the disorder. The spins interact via a pairwise potential J depending only on the distance between the spins. The Hamiltonian is the one written down in the introduction,

$$H = -\tfrac{1}{2}\sum_{ij} J(r_i - r_j) S_i S_j. \tag{2.1}$$

We shall proceed in the derivation assuming that $J(r)$ is positive, thus giving rise to ferromagnetic interactions. Later, we shall also discuss purely antiferromagnetic interactions, and at stages in the development will point out the sign changes necessary for a negative potential. A Hubbard–Stratonovich transformation expresses the partition function as

$$Z_N = \sum_{S_i} \int \mathcal{D}\phi [\det J\beta]^{\frac{1}{2}} \exp\left(-\frac{1}{2\beta} \iint \phi(r) J^{-1}(r - r')\phi(r')\,\mathrm{d}r\,\mathrm{d}r' + \sum_{i}^{N} \phi(r_i) S_i \right). \tag{2.2}$$

Employing replicas, we average out the site disorder by integrating over the positions r_i using the flat measure $:\frac{1}{V^N}\int_V \Pi \mathrm{d}r_i$.

$$\bar{Z}^n = \int \mathcal{D}\phi_a [\det J\beta]^{-\frac{n}{2}} \exp\left(-\frac{1}{2\beta} \iint \sum_a \phi_a(r) J^{-1}(r - r')\phi_a(r')\,\mathrm{d}r\,\mathrm{d}r' + N \log \frac{1}{V} \int \mathrm{Tr} \exp\left(\sum_a \phi_a(r) S_a \right) \mathrm{d}r \right). \tag{2.3}$$

Where we have introduced the trace over single-site spins S_a as convenient notation rather than write explicit $\cosh \phi_a$ factors. A field theoretical analysis of the above theory is complicated by the presence of the log term in the action. We overcome this difficulty by making a physically desirable modification to the definition of the disorder. In general one might expect the system to have been taken from a much larger system with a mean concentration of spins per unit volume, ρ. A suitably large subsystem of volume V will thus contain a number of spins N which is random and Poisson distributed: $P(N) = \mathrm{e}^{-\rho V} \frac{(\rho V)^N}{N!}$. This distribution must be used to weight the averaged free energy,

$$\begin{aligned} -\beta F_\Xi &= -\beta \sum_N P(N) \bar{F}_N = \sum_N P(N) \overline{\log Z_N} \\ &= \lim_{n \to 0} \sum_N P(N) \left(\frac{\bar{Z}_N^n - 1}{n} \right) = \lim_{n \to 0} \left(\frac{\Xi^n - 1}{n} \right) \end{aligned} \tag{2.4}$$

so we are led to define the partition function, $\Xi^n = \Sigma_N P(N) \bar{Z}_N^n$. By analogy with the statistical mechanics of pure systems, we shall call this type of disorder 'grand canonical disorder'.

The theory is defined by the partition function for grand canonical disorder:

$$\Xi^n = \mathrm{e}^{-\rho V} \int \mathcal{D}\phi_a [\det J\beta]^{-\frac{n}{2}} \exp\left(-\frac{1}{2\beta} \iint \sum_a \phi_a(r) J^{-1}(r-r')\phi_a(r')\,\mathrm{d}r\,\mathrm{d}r' + \rho \int \mathrm{Tr}\exp\left(\sum_a \phi_a(r) S_a\right) \mathrm{d}r \right). \tag{2.5}$$

Expanding the trace one sees that the leading term mixing replicas corresponds to a random temperature or random mass, familiar from the bond-disordered approaches, and that depending on the choice of interaction one might expect similar renormalization group results [12, 9]. The simplicity of this form is evident, but the physical content is not immediately obvious so we now turn to a discussion of this point.

2.1. Physical operators

In order to relate this theory to measurable quantities we must understand the type of excitations occurring in the theory and identify them with physical operators. This is not a simple task since the bound states of (2.5) are certainly not apparent at first glance.

We start by returning to the original formulation of the model and considering the operator most closely associated with the field ϕ_a appearing in the theory. This is the spin-density operator,

$$M_a(r) = \sum_i \delta(r - r_i) S_i^a \tag{2.6}$$

which appears in the replicated but unaveraged action as a source for ϕ_a.

$$Z^n = \int \mathcal{D}\phi_a [\det J\beta]^{-\frac{n}{2}} \exp\left(-\frac{1}{2\beta} \iint \sum_a \phi_a J^{-1} \phi_a \,\mathrm{d}r\,\mathrm{d}r' + \int \sum_a M_a \phi_a \,\mathrm{d}r \right). \tag{2.7}$$

The equations of motion (or Ward identities) at this unaveraged level provide relations between expectation values of M_a's and ϕ_a's. The equations of motion constitute a powerful tool and continue to hold for the fully averaged theory. For example the simplest relation is $\langle M_a \rangle = \langle \phi_a \rangle / \beta \tilde{J}(0)$. To make a connection with measurements performed on a particular sample with given disorder, we must look at self-averaging quantities. One such quantity is the magnetization density, M, and below we show how it is related to the averaged theory, the replicated theory and finally to the expectation values of fields ϕ_a.

$$M \stackrel{large\ V}{\longrightarrow} \frac{1}{V} \int \mathrm{d}r\, \overline{\langle M(r) \rangle} = \rho [\langle S \rangle]_{av} = \lim_{n\to 0} \frac{1}{V} \sum_n \int \mathrm{d}r\, \langle M_a(r) \rangle = \frac{\langle \phi \rangle}{\beta \tilde{J}(0)}. \tag{2.8}$$

The over-line denotes the average over the full grand canonical disorder. In the third expression we have inserted the definition of the magnetization-density operator to show the relation to the spins, and in this case the square brackets indicate the average over site disorder alone. In the last term we have assumed that $\langle \phi_a \rangle$ is independent of the replica index.

Similar arguments can be applied to the correlation function of magnetization-density operators. First, $\langle M_a(r) M_b(r') \rangle$, is related to the field correlator $\langle \phi_a(r) \phi_b(r') \rangle$, by,

$$\langle \tilde{M}_a(k) \tilde{M}_b(-k) \rangle = \frac{\langle \tilde{\phi}_a(k) \tilde{\phi}_b(-k) \rangle}{\beta^2 \tilde{J}^2(k)} - \frac{\delta^{ab}}{\beta \tilde{J}(k)} \tag{2.9}$$

where we have assumed spatial translation invariance. In an experiment, the neutron elastic scattering cross section is proportional to the following correlator,

$$\overline{\langle \tilde{M}(k)\tilde{M}(-k)\rangle} = \rho^2[\langle \tilde{S}(k)\tilde{S}(-k)\rangle]_{av} = \lim_{n\to 0}\frac{1}{n}\sum_a \langle \tilde{M}_a(k)\tilde{M}_a(-k)\rangle. \qquad (2.10)$$

The connected and disconnected parts of this correlation function can also be determined separately.

The spin-density operator we have considered so far is a simple generalization within the grand canonical context of the familiar spin operator. The expectation value, M in (2.8), is the order parameter for ferromagnetism, but $M_a(r)$ is certainly not the operator sensitive to spin-glass ordering. To probe this aspect of the physics it is natural to consider another operator:

$$q_{ab}(r) = \sum_i \delta(r - r_i) S_i^a S_i^b. \qquad (2.11)$$

This operator is familiar from spin-glass physics, its expectation value is an order parameter for the transition and its correlator is related to the nonlinear susceptibility. In fact, bond-disordered models such as the Edwards–Anderson model lead to field theories [13] in which the basic field variable is $q_{ab}(r)$ itself. These models immediately lead to non-trivial mean field theories describing the spin-glass order parameter. The theory (2.5), on the other hand, contains only the simple fields ϕ_a so the spin-glass physics is hidden in the composite operators (2.11) which do not appear at mean field level and necessarily require a deeper understanding of the field theory. In general, we cannot provide such a simple and general relation between expectation values of $q_{ab}(r)$ and of the fields ϕ_a as we were able to for the spin densities. However, within the context of the variational approximation, we will see that using linear response we can derive expressions for $\langle q_{ab}(r)\rangle$ and equations obeyed by the correlators $\langle \tilde{q}_{ab}(k)\tilde{q}_{cd}(-k)\rangle$. The connection between $\langle q_{ab}(r)\rangle$ and the usual physical order parameter of Edwards and Anderson [2] is clear from the relation $\langle q_{ab}(r)\rangle = \rho[\langle S\rangle^2]_{av}$. Although physical observables do not directly measure the correlators, these functions form a central part of our understanding in finite dimensions of both bond-disordered spin glasses and the site-disordered systems considered here. We shall concentrate our attention on the connected part of the spin correlator which is related to the q_{ab} correlators as follows:

$$\rho^2[\langle \tilde{S}(k)\tilde{S}(-k)\rangle^2_{con}]_{av} = \lim_{n\to 0}\frac{1}{n(n-1)}\sum_{a\neq b}\left(\langle \tilde{q}_{ab}(k)\tilde{q}_{ab}(-k)\rangle - \langle \tilde{q}_{aa}(k)\tilde{q}_{bb}(-k)\rangle\right). \qquad (2.12)$$

More detailed information about the theory could be deduced by studying higher moments of the spin probability distribution. To do this, operators involving more spins can be introduced in a similar way.

3. Variational method

As we have already argued, mean field theory does not describe the interesting physics of this model and one must consider some improvement. In this section we analyse the theory using the Gaussian variational method which will allow us to substantially develop the theory without making an explicit choice for the form of the interaction $J(r)$. The Gaussian variational equations are truncations of the full Schwinger–Dyson equations and become exact in the limit of many spin components (such an m-component theory is treated in a separate publication [5]). We find it convenient to phrase the development in terms of a variational method which allows us to calculate thermodynamic quantities simply, but from the outset we would like to acknowledge the difficulties of variational methods for field theory [14]. In the context of disordered systems, this method has had success in calculating exponents for random manifolds [15], and earlier in the problem of random heteropolymers [16]. On the other hand, it has been less successful in situations where RSB occurs in a one-step pattern. One should also bear in mind that important effects may occur at higher orders in $1/m$ and indeed we shall discover that the approximation does not obey certain requirements expected on general grounds for purely ferromagnetic interactions.

From the variational point of view, one selects a trial Hamiltonian, H_t, and the method may be motivated as follows:

$$\begin{aligned} \mathrm{e}^{-n\beta F_\Xi} &= \Xi^n = \mathrm{e}^{-\rho V} \int \mathcal{D}\phi \mathrm{e}^{-(H-H_t)-H_t} = \langle \mathrm{e}^{-(H-H_t)} \rangle_t \mathrm{e}^{-F_t} \mathrm{e}^{-\rho V} \\ &> \mathrm{e}^{-\langle (H-H_t) \rangle_t} \mathrm{e}^{-F_t} \mathrm{e}^{-\rho V}. \end{aligned} \tag{3.1}$$

So the variational free energy, $n\beta F_{var} = F_{trial} + \langle H - H_{trial} \rangle_{trial} + \rho V$, provides a bound to the true free energy: $F_\Xi < F_{var}$. When the replica limit is taken, this bound is no longer rigorous; nevertheless, the expression for the variational free energy is still valid. The Gaussian variational method simply consists in making a Gaussian trial, that is, choosing a trial Hamiltonian that is quadratic. Mean field theory can also be viewed as variational, in which case it corresponds to a linear trial Hamiltonian, and in this sense the Gaussian variational method is a simple generalization of mean field theory.

We allow the possibility of ferromagnetic order and make the following Gaussian ansatz (in which we have assumed translational invariance),

$$H_{trial} = \tfrac{1}{2} \int \sum_{ab} (\phi_a(r) - \bar{\phi}_a) G^{-1}_{ab}(r - r')(\phi_b(r') - \bar{\phi}_b)\, \mathrm{d}r\, \mathrm{d}r'. \tag{3.2}$$

The variational parameters, $\bar{\phi}_a$ and $G_{ab}(r)$ are simply related to ϕ expectation values: $\langle \phi_a \rangle = \bar{\phi}_a$ and $\langle \phi_a(r)\phi_b(r') \rangle = G_{ab}(r - r')$. This gives rise to a variational free energy of the following form,

$$\begin{aligned} n\beta F_{var} = &-\frac{1}{2} \operatorname{Tr} \log G_{ab} + \frac{1}{2\beta} \iint \sum_a (\bar{\phi}_a J^{-1}(r - r')\bar{\phi}_a + G_{aa}(r - r')J^{-1}(r - r'))\, \mathrm{d}r\, \mathrm{d}r' \\ &-\rho \operatorname{Tr} \int \exp\left(\sum_a \bar{\phi}_a S_a + \tfrac{1}{2} \sum_{ab} G_{ab}(r, r) S_a S_b \right) \mathrm{d}r \\ &+\frac{n}{2} \operatorname{Tr} \log(\beta J - 1) + \rho V \\ = &-\frac{V}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \sum_a [\log \tilde{G}(k)]^{aa} + \frac{V}{2\beta} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \sum_a \tilde{G}_{aa}(k) \tilde{J}^{-1}(k) \\ &+\frac{V}{2\beta} \sum_a \bar{\phi}_a^2 \tilde{J}^{-1}(0) - \rho V(\Omega - 1) + \frac{nV}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} (\log \beta J - 1). \end{aligned} \tag{3.3}$$

We have kept all the constant terms and have defined,

$$\Omega = \operatorname{Tr} \mathrm{e}^{H'} = \sum_{S_a = \pm 1} \exp\left(\sum_a \bar{\phi}_a S_a + \tfrac{1}{2} \sum_{ab} G_{ab}(0) S_a S_b \right). \tag{3.4}$$

The variational equations follow immediately by varying with respect to $\bar{\phi}_a$ and G_{ab}. Their general form is:

$$\begin{aligned} \bar{\phi}_a J^{-1}(0) &= \rho\beta\Omega_a = \rho\beta \operatorname{Tr} S_a \mathrm{e}^{H'} \\ \tilde{G}^{-1}_{ab} &= \frac{1}{\beta} \delta_{ab} \tilde{J}^{-1} - \rho\Omega_{ab} \end{aligned} \tag{3.5}$$

where we have introduced generalizations of (3.4):

$$\Omega_a = \operatorname{Tr} S_a e^{H'} = \sum_{S_a=\pm 1} S_a \exp\left(\sum_a \bar{\phi}_a S_a + \tfrac{1}{2}\sum_{ab} G_{ab}(0) S_a S_b\right)$$
$$\Omega_{ab} = \operatorname{Tr} S_a S_b e^{H'} = \sum_{S_a=\pm 1} S_a S_b \exp\left(\sum_a \bar{\phi}_a S_a + \tfrac{1}{2}\sum_{ab} G_{ab}(0) S_a S_b\right). \tag{3.6}$$

Further generalizations involving more indices will appear in future equations.

The variational equations, (3.5), can be interpreted diagrammatically as the leading equations of Schwinger–Dyson, truncated in at the level of the four-point function. As usual in this type of calculation, we find that Ω_{ab} is momentum independent; in many applications this only leads to mass renormalization, but due to the replica limit non-trivial effects can occur [15]. Some similarities exist with the formalism of [3]. The next few sections will be devoted to solving these equations assuming various forms for the replica structure.

The above derivation is for a ferromagnetic potential, in general the interaction potential is not positive definite and to deal with it correctly one must split up the k-space range accordingly. However, there is a simple case, that of a purely negative or antiferromagnetic interaction. There will be no magnetization, $\bar{\phi} = 0$ in this case, and at the level of the free energy, the effect is a sign change of G_{ab} in the definition of Ω (3.4). This leads to a reversed sign in the second variational equation and to a similar redefinition of Ω_{ab}.

Using standard thermodynamic relationships we can also determine the entropy and internal energy from the free energy (3.3),

$$\frac{nS}{V} = \frac{1}{2}\int \frac{d^d k}{(2\pi)^d} \sum_a [\log \tilde{G}(k)]^{aa} - \frac{1}{\beta}\int \frac{d^d k}{(2\pi)^d} \sum_a \tilde{G}_{aa}(k)\tilde{J}^{-1}(k)$$
$$- \frac{1}{\beta}\sum_a \bar{\phi}_a^2 \tilde{J}^{-1}(0) + \rho(\Omega - 1) + \frac{n}{2}\int \frac{d^d k}{(2\pi)^d}(2 - \log \beta J) \tag{3.7}$$
$$\frac{nU}{V} = -\frac{1}{2\beta^2}\int \frac{d^d k}{(2\pi)^d} \sum_a \tilde{G}_{aa}(k)\tilde{J}^{-1}(k) - \frac{1}{2\beta^2}\sum_a \bar{\phi}_a^2 \tilde{J}^{-1}(0) + \frac{n}{2\beta}\int \frac{d^d k}{(2\pi)^d}.$$

At high temperature, in the RS phase with zero magnetization, these formulae simplify considerably. We find, $S = \rho V \log 2$ as expected for Ising spins, and $U = -\rho J(0)/2$ because in this limit the only spin correlation is from the same spin term of (2.1).

3.1. Physical operators in the Gaussian approximation

In the context of this approximation we can calculate expectation values of products of the physical operators identified in section 2.1. In general for variational theories these quantities should be determined using linear response as this leads, in the variational sense, to a smaller error than a direct evaluation [17]. At the level of $\bar{Z}^n$ (2.7), we introduce suitable sources for the operators of interest, follow through the analysis to obtain the generalized variational free energy, and then take appropriate derivatives before setting the sources to zero. Sources for the operators $M_a(r)$ and $q_{ab}(r)$, only modify the free energy (3.3) through Ω, which becomes,

$$\Omega(r) = \operatorname{Tr} e^{H'(r)} = \operatorname{Tr}\exp\left(\sum_a (\bar{\phi}_a(r) + \beta h_a(r)) S_a + \tfrac{1}{2}\sum_{ab}(G_{ab}(r,r) + \beta j_{ab}(r) S_a S_b)\right) \tag{3.8}$$

where $h_a(r)$ and $j_{ab}(r)$ are the respective sources for $M_a(r)$ and $q_{ab}(r)$. Translational invariance is lost until the sources are set to zero at the end of the calculation.

To illustrate the method, consider the spin density operator $M_a(r)$. The magnetization is defined as:

$$M_a(r) = -\frac{1}{\beta}\frac{\delta F_{var}}{\delta h_a(r)} = \rho\Omega_a(r). \tag{3.9}$$

Using the variational equation of motion for $\bar{\phi}$ (3.5) we recover the equation $M_a(r) = \frac{1}{\beta}\tilde{J}^{-1}\bar{\phi}_a$ that we used in (2.8). In fact, this method merely reproduces the equations of motion we had earlier for expectation values of spin-density operators, and the correlation function is given by (2.9).

For the spin-glass operator, $q_{ab}(r)$, there is no alternative method and in this case applying linear response gives:

$$\begin{aligned}\langle q_{ab}(r)\rangle &= -\frac{1}{\beta}\frac{\delta F_{var}}{\delta j_{ab}(r)} = \rho\Omega_{ab}(r)\\ Q_{abcd}(r-r') = \langle q_{ab}(r)q_{cd}(r')\rangle &= -\frac{1}{\beta^2}\frac{\delta^2 F_{var}}{\delta j_{ab}(r)\delta j_{cd}(r')} = \frac{\rho}{\beta}\frac{\delta\Omega_{ab}(r)}{\delta j_{cd}(r')}.\end{aligned} \tag{3.10}$$

Using the variational equation of motion, some appropriate inverses and setting magnetizations to zero we finally obtain the equations:

$$\begin{aligned}\tilde{Q}_{abcd}(k) &= \rho\Omega_{abcd} + \frac{\rho}{2}\sum_{gh}\tilde{\Sigma}_{abgh}(k)\tilde{Q}_{ghcd}(k)\\ \tilde{\Sigma}_{abgh}(k) &= \sum_{ef}\Omega_{abef}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\tilde{G}_{eg}(p)\tilde{G}_{fh}(k-p).\end{aligned} \tag{3.11}$$

Where Ω_{abcd} is a trace of the form (3.4) containing four spins. This equation goes beyond the simple Gaussian approximation, as is clear from its diagrammatic interpretation. It may be helpful for intuition to note that a similar diagram appears in the BCS theory of superconductivity. One reason for the importance of being able to get explicit expressions for the spin-glass correlators is their use in showing the breakdown of the approximation for ferromagnetic interactions in section (4.2).

Equations of this type, involving objects with four replica indices are familiar in the theory of spin glasses. In this particular case, note that although symmetry in $a \leftrightarrow b$ and $c \leftrightarrow d$ is manifest, the symmetry between pairs $(ab) \leftrightarrow (cd)$ is not. Also remember that, in contrast to the SK mean field situation, diagonal terms with two equal indices do not vanish. In appendix A we solve this equation in the RS case but in the case of continuous RSB an extension of the methods of [18] would be needed.

4. Replica symmetric case

We now specialize to RS solutions of the equations. Our parametrization of G_{ab} is: $g_0 + g_1$ on the diagonal and, g_1 elsewhere. The constant term $\bar{\phi}_a$ is taken to be replica independent. In this RS case we may write an explicit expression for Ω (3.4), which leads to the following form (with ferromagnetic signs) for the variational free energy,

$$\begin{aligned}\beta\frac{F_{var}}{V} = &-\frac{1}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\left(\log\tilde{g}_0 + \frac{\tilde{g}_1}{\tilde{g}_0}\right) + \frac{1}{2\beta}\int\frac{\mathrm{d}^d k}{(2\pi)^d}(\tilde{g}_0+\tilde{g}_1)\tilde{J}^{-1} + \frac{1}{2\beta}\bar{\phi}^2\tilde{J}^{-1}(0)\\ &-\frac{\rho}{2}g_0(0) - \frac{\rho}{\sqrt{2\pi}}\int\mathrm{d}\xi\,\mathrm{e}^{-\frac{\xi^2}{2}}\log\left(2\cosh\left(\bar{\phi}+\xi\sqrt{g_1(0)}\right)\right)\end{aligned}$$

$$+\frac{1}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}(\log\beta J-1) \tag{4.1}$$

from which the RS version of the variational equations may be read off.

According to the first of the equations (3.10), we define the Edwards–Anderson order parameter q as Ω_{ab} for $a \neq b$. This order parameter takes values in $[0, 1]$ and is given by:

$$q=\Omega_{a\neq b}=\frac{1}{\sqrt{2\pi}}\int\mathrm{d}\xi\,\mathrm{e}^{-\frac{\xi^2}{2}}\tanh^2\left(\bar{\phi}+\xi\sqrt{g_1(0)}\right). \tag{4.2}$$

The variational equations for the components of the correlator are simply solved to give,

$$\begin{aligned}\tilde{g}_0(k)&=\frac{\beta\tilde{J}(k)}{1-(1-q)\rho\beta\tilde{J}(k)}\\ \tilde{g}_1(k)&=\rho q\tilde{g}_0^2(k)=\frac{\rho q\beta^2\tilde{J}^2(k)}{(1-(1-q)\rho\beta\tilde{J}(k))^2}.\end{aligned} \tag{4.3}$$

The value of the element g_1 at zero spatial distance, $g_1(0)$, appears throughout the equations and can be determined from the momentum space form above as,

$$g_1(0)=\rho q\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\beta^2\tilde{J}^2(k)}{(1-(1-q)\rho\beta\tilde{J}(k))^2}. \tag{4.4}$$

The equation for $\bar{\phi}$ is simply re-expressed in terms of the magnetization $M=\bar{\phi}/\beta\tilde{J}(0)$ (2.8). Together with the expression for q, we obtain a pair of equations defining the two order parameters and which bear a striking resemblance to the RS mean field equations for the Sherrington–Kirkpatrick model [1],

$$\begin{aligned}M&=\frac{\rho}{\sqrt{2\pi}}\int\mathrm{d}\xi\,\mathrm{e}^{-\frac{\xi^2}{2}}\tanh\left(\beta\tilde{J}(0)M+\xi\sqrt{g_1(0)}\right)\\ q&=\frac{1}{\sqrt{2\pi}}\int\mathrm{d}\xi\,\mathrm{e}^{-\frac{\xi^2}{2}}\tanh^2\left(\beta\tilde{J}(0)M+\xi\sqrt{g_1(0)}\right).\end{aligned} \tag{4.5}$$

In general a numerical solution is necessary, but at high temperature and low density a unique $q = 0$, $M = 0$ paramagnetic solution is expected. Depending on the interaction potential, which may or may not allow a ferromagnetic state, the solution leads to two types of critical line corresponding to the order parameters M of ferromagnetism and to q of spin-glass order which divide the phase diagram in the temperature density plane. These solutions, however, may be either unstable or be energetically unfavourable with respect to solutions in which replica symmetry is broken. To analyse these issues in detail requires a choice of potential and a numerical solution, which we give for certain examples in sections 6 and 7. Meanwhile we discuss the general conclusions that can be drawn from the simple $q = 0$, $M = 0$ solutions, the stability criteria and issues that arise if there is ferromagnetic order.

Given a solution of (4.5) for q and M, the correlators for the spin-density operators can simply be determined by (2.9) using the variational form: $\langle\phi_a(r)\phi_b(r')\rangle = G_{ab}$. The correlator for the q_{ab}'s is less immediate and in appendix A we give the general RS solution of (3.11).

4.1. Region with $q = 0$, $M = 0$

The M equation of (4.5) can always be satisfied by setting $M = 0$, and if a solution of the q equation exists then there is a paramagnetic phase. A particularly simple situation occurs in the region of high temperature and low density where $q = 0$ is a unique solution for a wide class of potentials and dimensions.

In this case, g_1 vanishes and the ϕ_a propagator is diagonal in replica indices. It follows from (2.9) that the magnetization correlator is also diagonal. By using the results of appendix A for $q = 0$, or by noticing that for diagonal G_{ab}, the second equation (3.11) simplifies to give $\tilde{\Sigma}_{abcd} = \Omega_{abcd} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \tilde{g}_0(p)\tilde{g}_0(k-p)$, the q_{ab} correlator can also be found,

$$\begin{aligned} \langle \tilde{M}_a(k)\tilde{M}_a(-k)\rangle &= \frac{\rho}{(1-\rho\beta\tilde{J}(k))} \\ \langle \tilde{q}_{ab}(k)\tilde{q}_{ab}(-k)\rangle &= \frac{\rho}{(1-\rho\int \frac{\mathrm{d}^d p}{(2\pi)^d}\tilde{g}_0(p)\tilde{g}_0(k-p))}. \end{aligned} \tag{4.6}$$

All other q_{ab} correlations vanish except for $\langle \tilde{q}_{aa}(k)\tilde{q}_{bb}(-k)\rangle = \rho$. The spin correlators follow simply from these formulae according to (2.10) and (2.12).

Both correlators may, depending on the type of interaction, diverge somewhere in the temperature-density plane. As temperature is lowered, the divergence in $[\langle \tilde{S}(k)\tilde{S}(-k)\rangle^2]_{av}$ will always occur before (or at the same point) as the divergence in $[\langle \tilde{S}(k)\tilde{S}(-k)\rangle]_{av}$. If we assume that the first divergence occurs at zero momentum, and can therefore correspond to a continuous spin-glass phase transition, the line on which it occurs is given by the condition:

$$\rho \int \frac{\mathrm{d}^d k}{(2\pi)^d}\tilde{g}_0(k)\tilde{g}_0(k) = \frac{g_1}{q} = 1. \tag{4.7}$$

Where we have used the relation $\tilde{g}_1 = \rho q \tilde{g}_0^2$ (4.3), which shows that g_1 is linear in q. The spin-density correlator generally remains finite on this line. The above condition is identical to the condition for the first appearance of a non-zero q solution in the equations (4.5), since by expanding the right-hand side of the q equation (for $M = 0$) we have,

$$q = g_1 - 2g_1^2 + \tfrac{17}{3}g_1^3 \ldots. \tag{4.8}$$

The coincidence of the two lines is not surprising as it only relates to two different ways of identifying the line of spin-glass transition.

In appendix B we present the usual stability analysis along the lines of the work by de Almeida and Thouless [19], and find the following condition that the replicon eigenvalue remains positive,

$$\rho r \int \frac{\mathrm{d}^d k}{(2\pi)^d}\tilde{g}_0^2 = \frac{r}{q} g_1 < 1. \tag{4.9}$$

Where r is given by

$$r = (1 - 2\Omega_{bc} + \Omega_{abcd}) = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}\xi\, \mathrm{e}^{-\frac{\xi^2}{2}} \cosh^{-4}\left(\beta\tilde{J}(0)M + \xi\sqrt{g_1(0)}\right). \tag{4.10}$$

The solution discussed above, in which q vanishes, has $r = 1$, and is only stable above the same spin-glass transition line, now also identified as the AT line. Just below the AT line, the perturbative RS solution with q small is always unstable. Performing the same expansion as in (4.8) for small g_1 with $M = 0$ for the r equation above, we find $r = 1 - 2g_1 + 7g_1^2 + \mathrm{O}(g_1^3)$. The condition (4.9) is, therefore, always violated and the AT line signals a non-trivial breaking of replica symmetry. This indicates that we have a situation similar to that found in the SK model: namely a high temperature $q = 0$ phase separated from a spin-glass phase with non-trivial RSB. Usually the coincidence of lines noted here indicates that the RSB will take place continuously. The reader will doubtless note that the above analysis seems remarkably general both in terms of the type of interaction involved and on the dimensionality of the space. One sees that when presented with the statistics of the matrix $J(r_i - r_j)$ it is very difficult to divine from what interaction type and what dimension of space it arises. The consideration of a genuine cut-off interaction where one would expect the geometry and type of interaction to play a much stronger role is relegated to a future study.

4.2. Magnetization

For a ferromagnetic interaction the mean field equations, $M = \rho \tanh(\beta \tilde{J}(0) M)$ suggest that we should expect a ferromagnetic transition at $\rho\beta\tilde{J}(0) = 1$. The Gaussian variational approximation sees this transition as the divergence in the M_a correlator in (4.6). However, it is clear from the preceding discussion that the spin-glass transition intervenes and that we should not even be attempting to solve the RS equations (4.5) to locate a ferromagnetic transition because the system is already in an RSB phase. The AT line shrouds the critical region and the RS equations are not relevant to the transition.

Such a phenomenon may be surprising and the order in which the transitions occur deserves further discussion. For a generic interaction and generic number of spin components it is reasonable for the spin-glass transition to occur at higher temperature than the ferromagnetic one. For example this is known to happen in the random field Ising model, where replica symmetry breaking has been shown to take place before the ferromagnetic transition [8]. The problem occurs for Ising spins and a purely ferromagnetic interaction (positive in space) where we might well expect the transitions to be concurrent.

Let us try to clarify this point† by noting that the divergence $[\langle \tilde{S}(k)\tilde{S}(-k)\rangle^2]_{av}$ occurs while $[\langle \tilde{S}(k)\tilde{S}(-k)\rangle]_{av}$ is still finite. The divergence implies that at sufficiently large spatial separation the spin-glass correlator is larger than the ferromagnetic one: $[\langle S(r)S(0)\rangle^2]_{av} > [\langle S(r)S(0)\rangle]_{av}$. We might expect that these correlators are distributed according to some probability distribution $P(C)$ and therefore write,

$$
\begin{aligned}
[\langle S(r)S(0)\rangle]_{av} &= \int P(C)C \, \mathrm{d}C \\
[\langle S(r)S(0)\rangle^2]_{av} &= \int P(C)C^2 \, \mathrm{d}C
\end{aligned}
. \tag{4.11}
$$

But for Ising spins we expect that the correlator $C = \langle S(r)S(0)\rangle$ is bounded, $0 \leqslant C \leqslant 1$, provided all the interactions are ferromagnetic. Using such bounds in the expressions above, it is clear that the inequality, $[\langle S(r)S(0)\rangle^2]_{av} \leqslant [\langle S(r)S(0)\rangle]_{av}$ holds for all r. This inequality is violated in our solutions. It should be possible to trace this breakdown of the Gaussian variational approximation to the neglect of some higher order (in number of spin components) term in the Schwinger–Dyson equations. Indeed, Sherrington [20] has pointed out the dangers inherent in methods similar to the one we use in the case of a Landau–Ginzburg approach to dilute ferromagnets, and has identified the relevant diagrams.

Even though it is inappropriate to look for a continuous ferromagnetic transition in the RS equations, it is interesting to see what happens. The usual argument, based on the graphical solution of (4.5), provides a simple condition on the gradient of the magnetization equation (at $M = 0$) for the existence of small M solutions: $(1-q)\rho\beta\tilde{J}(0) > 1$ (the $(1-q)$ term causes a suppression of the transition temperature with respect to the mean field prediction). On the other hand, for the Gaussian integrals to be well defined the propagators must be positive leading to, $(1-q)\rho\beta\tilde{J}(k) < 1$, $\forall k$. In this sense the usual condition always fails, even in finite volume. This argument does not preclude RS solutions

† We thank G Parisi for bringing this argument to our attention.

with magnetization, it only indicates that the ferromagnetic transition itself is hidden, or possibly first order. Indeed for suitable ferromagnetic interactions numerical analysis of the equations allows us to find ferromagnetic solutions below the transition with no difficulty. At zero temperature general arguments would require such a solution for interactions that do not vanish beyond some range. Since this issue requires a choice of potential we will discuss it further when we look at examples.

5. Replica symmetric breaking

Since we have found a scenario similar to that of the Sherrington–Kirkpatrick model, in which the RS $q \neq 0$ solution is unstable as soon as it appears, we shall look for continuous replica symmetry broken solutions. We consider Parisi matrices in which the off-diagonal part of the matrix $\tilde{G}_{ab}(k)$ is parametrized by a continuous function $\tilde{g}(u,k)$ where $u \in [0,1]$, and the diagonal part is denoted by $\tilde{g}_D(k)$. The algebra of such matrices has been developed, and expressions for products, inverses, etc are given in the appendix of [15]. We shall follow the notation used in that paper for certain integral transforms that occur frequently. Although the similarity to the random manifold problems seems clear the sign of the potential is opposite and is related to a random manifold problem with imaginary noise.

For G_{ab} a Parisi matrix, Ω (3.4), is very similar to the free energy in the SK model. It is well known that this cannot be obtained in a closed form and a standard strategy is to work close to the transition line by expanding Ω up to a term of $\mathrm{O}(g^4)$ which in the SK model is the first term leading to a breaking of replica symmetry [21]. We shall assume that there is no magnetization in this calculation because the denominators of the magnetization correlator showed no tendency to diverge at the AT line. The expansion is, [13],

$$\frac{\Omega - 1}{n} \approx \tfrac{1}{2} g_D(0) - \tfrac{1}{4} \int_0^1 \left(g^2 + \tfrac{1}{6} g^4 - \frac{u}{3} g^3 - g \int_0^u g^2 \right) \mathrm{d}u \tag{5.1}$$

where g is the spatial function, $g(u,0)$, evaluated at zero distance. The remaining terms in the action are easily computed within the algebra of Parisi matrices. The variational equations are,

$$\begin{aligned} [\tilde{g}_D(k)]^{-1} &= +\rho\sigma_D(k) = \rho\left(\frac{\tilde{J}^{-1}}{\beta\rho} - 1\right) \\ [\tilde{g}(u,k)]^{-1} &= -\rho\sigma(u) = 2\rho\frac{\delta\Omega}{\delta\tilde{g}}. \end{aligned} \tag{5.2}$$

Where the notation $[\tilde{g}]^{-1}$ indicates the component of the inverse matrix which is also of Parisi form. In deriving these equations of motion it is important to take care of the signs; for example a minus sign appears in $\delta/\delta\tilde{g}(u,k)\,\mathrm{Tr}\log G = -[\tilde{g}(u,k)]^{-1}$, and the signs for σ have been chosen in order that the following equations for the inverses appear in the simplest form. For the antiferromagnetic sign interaction the definition (5.1) is changed and a sign appears in the second of the variational equations.

Defining the denominator combinations:

$$\begin{aligned} D_D(k) &= \sigma_D + \langle\sigma\rangle \\ D(u,k) &= \sigma_D + \langle\sigma\rangle + [\sigma](u) \end{aligned} \tag{5.3}$$

where following the notation of [15] the angle and square brackets denote $\langle\sigma\rangle \equiv \int_0^1 du\sigma(u)$

and $[\sigma](u) \equiv u\sigma(u) - \int_0^u dv\sigma(v)$. The equations can be inverted to find,

$$\begin{aligned}\tilde{g}_D(k) &= \frac{1}{\rho D_D(k)}\left(1 + \int_0^1 \frac{\mathrm{d}u}{u^2}\frac{[\sigma](u)}{D(u,k)} + \frac{\sigma(0)}{D_D(k)}\right)\\ \tilde{g}(u,k) &= \frac{1}{\rho D_D(k)}\left(\frac{[\sigma](u)}{uD(u,k)} + \int_0^u \frac{\mathrm{d}v}{v^2}\frac{[\sigma](v)}{D(v,k)} + \frac{\sigma(0)}{D_D(k)}\right).\end{aligned} \tag{5.4}$$

Taking the derivative with respect to u of the second of these equations leads to the simple relation,

$$\tilde{g}' = \frac{1}{\rho}\frac{\sigma'}{D^2} = -\frac{1}{u\rho}\frac{\mathrm{d}}{\mathrm{d}u}\frac{1}{D}. \tag{5.5}$$

where we have made use of $D' = [\sigma]' = u\sigma'$. Proceeding by differentiating the second equation of (5.2) with respect to u and using the expression for g' above, one obtains $\sigma' = 0$, or,

$$\left(1 + g^2 - ug - \int_u^1 g\right) = \rho\left(\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{D^2}\right)^{-1}. \tag{5.6}$$

Taking another derivative in some region where equation (5.6) holds we find:

$$g = \alpha(u)u = \frac{u}{2}\left(1 + 2\rho^2\frac{\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{D^3}}{(\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{D^2})^3}\right). \tag{5.7}$$

Substituting this result into the relation (5.5) we find that σ is determined by a nonlinear, but first-order, differential equation in u,

$$\frac{\mathrm{d}}{\mathrm{d}u}(u\alpha(u)) = -\frac{1}{u\rho}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}}{\mathrm{d}u}\frac{1}{D}. \tag{5.8}$$

We cannot go further without being more explicit about the form of the interaction and dimension. In general the function $\alpha(u)$ has a complicated dependence on $\sigma(u)$ and the differential equation will be difficult to solve, but based on the examples we have studied, we expect that the RSB pattern as shown in figure 1 to be generic. The initial part of $g(u, 0)$ is determined by a power series analysis of the nonlinear differential equation about the origin, and in all examples we have studied the leading term is linear. Above some breakpoint, u_0 close to the origin, $g(u, 0)$ becomes constant. For consistency with the original expansion of Ω we must work close to the AT line and must require that the $g(u, 0)$ is perturbatively small in deviations from that line. Other patterns could be envisaged, though we have not yet found any examples.

For certain choices of interaction or dimension the differential equation may simplify radically. For example in large enough dimensions (the critical dimension depends on the large momentum behaviour of the interaction; if $\tilde{J} \sim k^{-2}$, then $d_{crit} = 4$) the second term in α will disappear in the limit in which the short-distance cut-off is removed. The resulting equation is simple and we find a scenario very similar to that found in the SK model. Due to a remarkable cancellation α is also a constant for Yukawa interactions in three dimensions.

This analysis near to the AT line is insufficient for some purposes such as to investigate continuous RSB solutions with magnetization, and one would like to solve the equations in greater generality. More terms could be kept in the expansion (5.1) [13, 22] or it may be possible to use the differential equation obeyed by Ω along the lines of the SK case [23]. Further difficulties would occur if it became necessary to consider the order parameters with more replica indices that can be constructed as in section 2.1 [24]. We do not attempt such generalizations here.

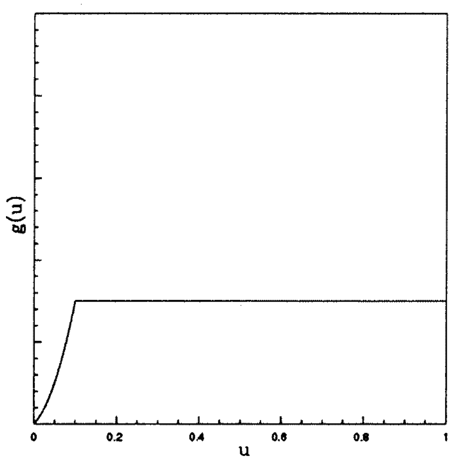

Figure 1. Typical RSB pattern just below the AT line.

We have also studied the equations for one step of RSB. We do not expect such solutions to be relevant near the AT line, and at lower temperatures their importance should be judged relative to some continuous solution. Another use of such solutions is in calculating the entropy of the meta-stable states which is relevant to the dynamics. We have made a calculation following Monasson [25] that demonstrates that as usual in SK-like situations, this entropy remains zero down to the AT line and that we therefore do not expect any distinct dynamical transition.

6. Yukawa potential

A simple potential for illustrative purposes is,

$$\tilde{J}(k) = \pm \frac{1}{\mu^{d-2}} \frac{1}{k^2 + \mu^2} \tag{6.1}$$

where the signs refer to ferromagnetic and antiferromagnetic interactions respectively. This momentum space form describes a Yukawa type potential that is screened on a length scale $1/\mu$. We choose to measure distances in terms of this scale (for example the dimensionless density becomes ρ/μ^d), and therefore set $\mu = 1$ in what follows.

The ferromagnetic sign leads to a rich phase diagram and the complications discussed in section 4.2. For this reason it is useful to consider the antiferromagnetic sign since antiferromagnetic order cannot exist in the absence of a lattice, and the phase diagram will be simpler, allowing us to concentrate on the properties of the spin-glass transition.

We discuss different dimensions separately, starting with three dimensions.

6.1. Three dimensions

In three dimensions the real space form of (6.1) is the well known Yukawa form,

$$J(r) = \pm \frac{e^{-\mu r}}{4\pi \mu r}. \tag{6.2}$$

In this case the integral (4.4) can be done to obtain g_1:

$$g_1 = \rho\beta^2 q \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2 + 1 \mp (1-q)\rho\beta)^2} = \frac{\rho\beta^2 q}{8\pi\sqrt{1 \mp (1-q)\rho\beta}} \tag{6.3}$$

which allows us to numerically solve the equations (4.5). At high temperature the unique solution is $q = 0$, $M = 0$, whereas at lower temperatures for the ferromagnet, the positivity of g_0 restricts the possible range of q to, $q > 1 - 1/\rho\beta$.

The spin-glass transition line is defined by the condition $8\pi\sqrt{1 \mp \beta\rho} - \rho\beta^2$, which can be solved to give,

$$\rho = 2(4\pi T)^2 \left(\mp T + \sqrt{T^2 + 1/(4\pi)^2}\right). \tag{6.4}$$

This is the only phase transition in the antiferromagnetic case whereas in the ferromagnet it lies very close to the line, $\rho\beta = 1$, where ferromagnetic ordering would occur in mean field approximation.

Above this line, in the high-temperature phase with $q = 0$, $M = 0$, only the diagonal magnetization correlator remains:

$$\langle M_a(r) M_a(0)\rangle = \rho\delta(r) + \frac{\rho^2 \beta e^{-r\sqrt{1-\rho\beta}}}{4\pi r}. \tag{6.5}$$

We have given the real space form because it has a simple interpretation. The first term is from the same spin, and the leading piece of the second term arises from a two-spin term. Only considering two spins, the correlator can be written in terms of Boltzmann factors which when expanded for small interaction strength give $\rho^2 \beta J(r)$. Using (6.2) one recovers the leading piece of the full correlator.

Still in the high-temperature region, we can also calculate the only non-trivial q_{ab} correlator for the P combination of indices (see appendix A):

$$\tilde{Q}_{abab}(k) = \langle \tilde{q}_{ab}(k)\tilde{q}_{ab}(-k)\rangle = \frac{\rho}{1 - \rho\int \frac{d^dk}{(2\pi)^3}\tilde{g}_0^2(k)} = \frac{\rho}{1 - \frac{\rho\beta^2}{4\pi}\frac{1}{k}\sin^{-1}\left(\frac{k}{\sqrt{4(1\mp\beta\rho)+2k^2}}\right)}. \tag{6.6}$$

The connected spin correlation function $[\langle \tilde{S}(k)\tilde{S}(-k)\rangle^2_{con}]_{av}$ is obtained by subtracting the term $\langle \tilde{q}_{aa}(k)\tilde{q}_{bb}(-k)\rangle = \rho$ (2.12). In the small momentum limit this expression simplifies and becomes proportional to $1/(k^2 + m^2)$ with mass given by,

$$m^2 = \frac{96\pi}{5\rho\beta^2}(1 - \beta\rho)\left(\sqrt{1 - \beta\rho} - \frac{\rho\beta^2}{8\pi}\right). \tag{6.7}$$

In accordance with the general analysis, criticality occurs for vanishing m^2 which is another way of finding the AT line. Despite the fact that we are employing a Hartree–Fock method, which in pure models can give non-classical exponents, here m^2 is an analytic expression and the exponent ν takes its mean field value of $\frac{1}{2}$.

At lower temperatures, other replica symmetric solutions exist. The simplest case is the antiferromagnet, for which there are two solutions below the AT line. One solution, $q = 0$, is unstable both with respect to longitudinal and replicon fluctuations, and the other has $q > 0$ and is numerically found to be unstable with respect to replicon fluctuations as expected from the general analysis in section 4.2. The ferromagnetic case is more complicated, beside $M = 0$ solutions (the $q = 0$ solution only exists for $\beta\rho \leqslant 1$) that are unstable just as for the antiferromagnet, there are solutions with $M \neq 0$. These magnetized solutions first appear on a second curve at slightly lower temperature than the AT line,

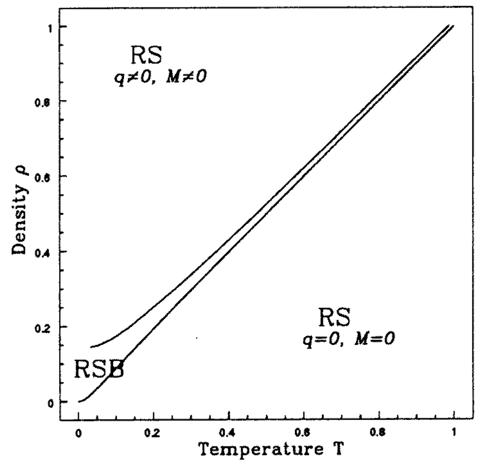

Figure 2. The regions in the density temperature plane where various RS solutions exist. For ferromagnetic Yukawa interactions in three dimensions. A region between the spin-glass and ferromagnetic transitions is unphysical.

as shown in figure 2. On this curve there is one new stable solution with non-vanishing magnetization and q, these values being largest at the low temperature end of the curve. As the temperature is reduced the solution splits into two; the branch with larger values of M and q remains stable and the magnetization continues to grow in the usual way, the other branch decreases in magnetization and becomes unstable with respect to the replicon not far below the curve shown on the figure. We suspect that the decreasing solution will not play any role.

The significance of the upper curve in figure 2 and the new solutions that it heralds depends on whether continuous solutions also exist in this region. Free energies must then be compared to identify a first-order transition. As was discussed in section 4.2, for purely ferromagnetic interactions we expect concurrent spin-glass and ferromagnetic transition curves, and therefore we are witnessing a shortcoming of the approximation. It seems likely that at sufficiently low temperature the single stable new RS ferromagnetic solution will have the lowest free energy and will represent the system. If this is the case, at zero temperature the transition must occur for finite density, suggesting a connection between the ferromagnetic and a percolation transition. This would be a reasonable conclusion for a ferromagnetic potential that strictly vanishes beyond some radius (it would be interesting to consider such a potential to clarify this connection). On the other hand, for an interaction of the type (6.2) which never vanishes, we should always expect a ferromagnetic RS ground state at zero temperature. Investigating this region in detail, we find that the upper curve shown in figure 2 does not quite extend down to zero temperature because the main magnetic solution becomes unstable with respect to replica fluctuation. This observation can be confirmed by making a low-temperature expansion of the RS equations (4.5). Apparently this is another failure of the approximation, and is maybe not surprising since it is unlikely that the $\beta \to \infty$ limit commutes with the large number of components limit used to justify the approximation.

We now turn to the form of continuous RSB just below the spin-glass transition. In view of the arguments showing that the critical region for the pure ferromagnet is not treated well by the approximation, we only discuss the antiferromagnet.

It is useful to identify mass parameters by writing the denominators (5.3) as,

$$\begin{aligned}
D_D(k) &= \frac{1}{\beta\rho}(k^2 + m_D^2) \\
m_D^2 &= (1 + \beta\rho(1 + \langle\sigma\rangle)) \\
D(u,k) &= \frac{1}{\beta\rho}(k^2 + m^2(u)) \\
m^2(u) &= m_D^2 + \beta\rho[\sigma](u).
\end{aligned} \tag{6.8}$$

Performing the momentum integrals over inverse powers of the denominator we find a remarkable cancellation so that the parameter α (5.7) is independent of u,

$$\alpha = \frac{1}{2}\left(\frac{32\pi^2}{\rho\beta^3} - 1\right). \tag{6.9}$$

We consider solutions consisting of a continuous piece up to a breakpoint u_0, followed by a constant as shown in figure 1. The differential equation is trivially solved to give $m(u) = m_D + 2\pi\alpha u^2/\beta$. Continuing to solve the RSB equations we find that in the continuous region, σ contains linear and cubic terms in u and that the breakpoint is determined by a quartic equation. Since u_0 is small we only give the leading terms,

$$\frac{\alpha}{\beta}(32\pi^2 \pm \rho\beta^3)u_0 = \left(\rho\beta^2 - 8\pi\sqrt{1+\rho\beta}\right). \tag{6.10}$$

The right-hand side vanishes on the AT line, recovering the RS solution with $u_0 = 0$. Expanding to first order about this line we find an expression for the breakpoint in perturbation theory,

$$u_0 = 4\frac{\delta\beta}{\beta}\frac{(\rho\beta^3)(\rho\beta^3 - 16\pi^2)}{(\rho\beta^3 - 32\pi^2)^2}. \tag{6.11}$$

This indicates that we have a consistent solution since g remains perturbatively small so the expansion near the AT line is justified. The magnetization correlator can be calculated in this region along the lines of (2.10), in which the sum now becomes an integral (5.4). To leading order we find

$$\tilde{g}_D(k) = \frac{\beta}{k^2 + m_D^2}\left(1 + \frac{4\pi\alpha m_D}{\beta(k^2 + m_D^2)}\right). \tag{6.12}$$

Where $m_D = \sqrt{1+\rho\beta} + \mathrm{O}(u_0)$.

These detailed predictions of a spin-glass phase for a site-disordered antiferromagnet are interesting because few analytic results exist for the problem. Preliminary numerical simulations of the dynamics [26] reach the same conclusion as those described by McLenaghan and Sherrington [27] (who also refer to experimental realizations); that is to say, no such phase is observed. More work is needed in this area since, for the analytic work, either the Gaussian variational approximation or the simple choice of a two-replica order parameter may be in doubt; whereas the numerical simulations suffer from small sizes and consequent dependence on the boundary conditions.

6.2. Yukawa in $d > 4$

In four or more dimensions many of the integrals diverge indicating dependence on details of an unknown short-distance theory. For example, in the RS phase the integral defining $g_1(0)$ diverges, the leading term (in the momentum cut-off Λ) gives:

$$g_1 = \rho\beta^2 q \frac{\Gamma_d}{(d-4)}\left(\frac{\Lambda}{\mu}\right)^{d-4}. \tag{6.13}$$

Where $\Gamma_d = \Omega_d/(2\pi)^d = 2/((4\pi)^{d/2}\Gamma(d/2))$. The AT line is thus given by,

$$\rho\beta^2 = \frac{(d-4)}{\Gamma_d}\left(\frac{\mu}{\Lambda}\right)^{d-4}. \tag{6.14}$$

So as the cut-off is removed ($\Lambda \to \infty$), the AT line moves towards the $\rho = 0$ axis, and the RSB phase covers the whole phase space. One can proceed to expand about this line to find a simple RSB with $\alpha = \frac{1}{2}$ and $u_0 = 4\delta\beta/\beta$.

Alternatively, we can recover a finite temperature transition by taking the infinite-dimension limit while scaling the cut-off as $\Lambda/\mu \sim \sqrt{d}$. This limit sets $g_1 \propto \beta^2 q J^2$ and yields the SK mean field equations.

6.3. Yukawa in $d < 3$

The two-dimensional example contains most of the features seen in low-dimensional Yukawa models, and we concentrate on it here. Again the integral can be done to obtain a simple form for g_1,

$$g_1 = \rho\beta^2 q \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{1}{(k^2 + 1 \mp (1-q)\rho\beta)^2} = \frac{\rho\beta^2 q}{4\pi(1 \mp (1-q)\rho\beta)}. \tag{6.15}$$

The AT condition is $4\pi(1 \mp \rho\beta) = \rho\beta^2$, so the line is given by:

$$\rho = \frac{4\pi T^2}{1 \pm 4\pi T}. \tag{6.16}$$

Note that in the antiferromagnetic case, the line always lies below $T = \frac{1}{4}\pi$. Numerical investigations of the RS equations lead to a picture qualitatively the same as the one described in the three-dimensional case. Quantitatively there are notable differences, in two dimensions the unphysical region between the AT line and the line on which stable ferromagnetic solutions first appear is wider than in figure 2.

In searching for a RSB solution for the antiferromagnetic case it is useful to work in terms of the mass parameters $m(u)$, defined by the combinations (6.8) rather than directly in terms of σ. In two dimensions we find $\alpha = \frac{1}{2}(\frac{16\pi^2}{\rho\beta^3}m^2 - 1)$. The differential equation (5.8) can be written in terms of the variable $x = u^2$, and it is possible to work in terms of $f = m^2$, to obtain,

$$\left(\frac{32\pi^2}{\rho\beta^3}xf - \frac{\beta}{\pi}\right)\frac{\mathrm{d}f}{\mathrm{d}x} + \left(\frac{16\pi^2}{\rho\beta^3}f - 1\right)f = 0. \tag{6.17}$$

Provided the breaking pattern is similar to that shown in figure 1 we need only analyse the small u behaviour of this equation. We find results very similar to the three-dimensional case: $m^2 \approx m_D^2(1 + \pi u^2(16\pi^2 m_D^2 - \rho\beta^3)/\rho\beta^4)$, and g approximately linear in u. This is a consistent procedure since near the AT line we obtain a sensible expression for the breakpoint,

$$u_0 = 4\delta\beta\frac{(\beta - 2\pi)}{(\beta - 4\pi)^2}. \tag{6.18}$$

Closer inspection of the differential equation shows that it is ill-behaved because solutions tend to blow up at finite values of the parameter u. Also note that two distinct, constant solutions exist, which may suggest a solution with one step of RSB.

One dimension is interesting because exact results are known about the bond disordered version of the model. The AT line is given at $4(1 \mp \rho\beta)^{3/2} = \rho\beta^2$, which is a cubic equation in ρ. A series solution of the differential equation shows behaviour like that of the two-dimensional case. The zero-dimensional case is formally similar to an RKKY potential in any dimension, and is better discussed in that context.

7. RKKY-like potential

The RKKY potential itself, which in three dimensions is $J(r) = \cos \mu r/r^3$, has a complicated Fourier transform. So as is usual in analytic work we consider instead:

$$\tilde{J}(k) = \begin{cases} \tilde{J}_0/\mu^d & \text{for } k < \mu \\ 0 & \text{for } k > \mu. \end{cases} \tag{7.1}$$

$J(r)$ oscillates in sign to have the correct qualitative form, but note that the negative parts are not very strong. As was the case with the Yukawa potential, μ and $\tilde{J}_0$ can be scaled away by measuring distances and temperatures approximately, so we will set $\mu = \tilde{J}_0 = 1$. The integral for g_1 can immediately be performed,

$$g_1 = \rho\beta^2 q \int_{|k|<\mu} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(1-(1-q)\rho\beta)^2} = \frac{\Gamma_d \rho\beta^2 q}{(1-(1-q)\rho\beta)^2} \tag{7.2}$$

where $\Gamma_d = \Omega_d/(2\pi)^d = 2/((4\pi)^{d/2}\Gamma(d/2))$, and in three dimensions, $\Gamma_3 = \frac{1}{6}\pi^2$.

The AT condition is given by

$$\frac{\Gamma_d \rho\beta^2}{(1-(1-q)\rho\beta)^2} = 1. \tag{7.3}$$

So at this transition the magnetization propagators remain finite. This can be solved to give the AT line,

$$\rho = \left(T + \frac{\Gamma_d}{2} - \sqrt{\Gamma_d T + \frac{\Gamma_d^2}{4}}\right). \tag{7.4}$$

Since the form of this potential does not depend strongly on the dimension we shall discuss only the three-dimensional case. Above the AT line, in the $q = 0$ RS phase, the correlations are

$$\langle \tilde{M}_a(k)\tilde{M}_a(-k)\rangle = \begin{cases} \dfrac{\rho}{1-\rho\beta} & \text{for } k < 1 \\ \rho & \text{for } k > 1 \end{cases} \tag{7.5}$$

and

$$\tilde{Q}_{(ab)(ab)}(k) = \begin{cases} \dfrac{\rho}{1 - \frac{\rho}{96\pi^2}(\frac{\beta}{1-\rho\beta})^2(k+4)(k-2)^2} & \text{for } k < 2 \\ \rho & \text{for } k > 2. \end{cases} \tag{7.6}$$

Where the non-trivial form of the second propagator arises from the slightly complicated integration region and this allows us to obtain a non-mean field exponent, $\eta = 1$.

Numerical study of the RS equations exposes a scenario very similar to the ferromagnetic Yukawa case. The solutions with zero magnetization are always unstable below the AT line, but new magnetized solutions appear on another line. We recognize the second line from the ferromagnetic experience, at very high densities the positive short range part of the potential dominates and the system magnetizes, returning to a RS form. The phase diagram is shown in figure 3.

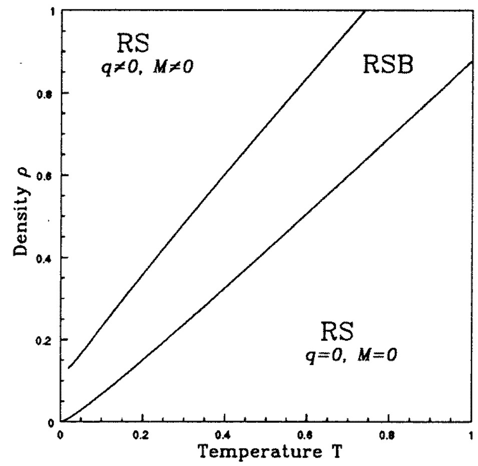

Figure 3. Regions of existence of RS solutions for RKKY-like interactions in three dimensions.

For RSB we find $\alpha = \frac{1}{2}(1 + \frac{2}{\Gamma_d^2 \rho\beta^3} m^6)$. The differential equation can be written in terms of $f = m^2$ where m is defined in a similar way to (6.8),

$$\left(\frac{12}{\Gamma_d^2 \rho\beta^3} x f^4 - 4\beta\Gamma_d \right) \frac{\mathrm{d}f}{\mathrm{d}x} + \left(\frac{2}{\Gamma_d^2 \rho\beta^3} f^3 + 1 \right) f^2 = 0. \tag{7.7}$$

The expression for the breakpoint is sensible and given by,

$$u_0 = 4\Gamma_d \delta\beta \frac{\rho\beta}{(1+\rho\beta)(1-\rho\beta)}. \tag{7.8}$$

From this information we can calculate the structure of the two-index correlators $\tilde{g}(k, u)$, leading to non-trivial momentum dependence in $\tilde{g}_D(k)$; however, the analysis only holds close to the spin-glass transition and cannot address the ferromagnetic transition itself. Indeed, at the ferromagnetic line, g will be large and the expansion of Ω is manifestly inappropriate. In addition the four-index correlation functions Q_{abcd} contain much of the physics of the spin-glass phase: for example the θ-exponent [7] may be extracted from the long-distance behaviour of such objects. Equation (3.11) is, however, an equation carrying four replica indices and the solution in the case of continuous replica symmetry breaking is technically rather formidable requiring extensions of the methods described in [18].

8. Conclusions

The Gaussian variational method, which is the simplest non-trivial way of analysing this site-disordered field theory, provides an interesting picture of the model. For all forms of interaction between the spins we find a spin-glass transition which separates a high-temperature paramagnetic phase from a low-temperature phase with non-vanishing Edwards–Anderson order parameter. The transition is signalled by RSB which takes place continuously just below the transition. We have carefully displayed the connection between physical observables and the replicated quantities we calculate and see explicitly how the spin-glass correlator, $\langle \tilde{q}_{ab}(k)\tilde{q}_{cd}(-k)\rangle$, diverges as the transition is approached from above.

The properties of the low-temperature region depend on the precise form of the interaction, but not sensitively on dimension in fewer than four dimensions. Our study of antiferromagnetic Yukawa interactions indicates that the low-temperature phase breaks replica symmetry continuously at all temperatures. The detailed form of the breaking has, however, only been determined in the vicinity of the transition. This prediction is interesting in view of the numerical simulations which contradict it and further work is needed in this case. In the case of ferromagnetic interactions, or the RKKY-like interactions which contain a substantial ferromagnetic component, we have found stable replica symmetric solutions with non-vanishing magnetization. It seems likely that at sufficiently low temperature one such solution will describe the system. We are unable to be certain on this point until we know about the existence and properties of solutions with continuous replica symmetry breaking in the region away from the spin-glass transition line. This problem also means that we are not presently able to directly analyse the ferromagnetic transition, though, because the magnetization correlators remain at the spin-glass transition, it is clear that the transitions are not coincident indicating a breakdown of the approximation in the case of a pure ferromagnet. Some further difficulties with the zero temperature limit have been mentioned in the text.

Besides the inaccessibility of the ferromagnetic transition, an important aspect of the spin-glass physics is missing. Namely, the behaviour of the spin-glass correlator below the transition where is should remain massless yet have a non-trivial exponent θ [7]. We have derived equations obeyed by the correlator, but because it is a four-index quantity, the technical difficulties of solving the equation in the continuously broken spin-glass phase are presently beyond us. Both the problems mentioned can be addressed in the case of m-component Heisenberg spins the limit $m \to \infty$[5].

The two issues in which the approximation is clearly unreliable are in its treatment of the purely ferromagnetic interaction and its relative insensitivity to dimension or interaction type. It therefore seems appropriate to add some further words on when we expect the approximation to be trustworthy. The Gaussian variational approximation can be seen as making a Gaussian ansatz on the Schwinger–Dyson equations of the theory for the fields $\phi_a(x)$ (in the case where there is RSB the ansatz is actually more refined in terms of nested Gaussians). However, this ansatz can only be justified in certain circumstances. Some form of the central limit theorem must be bought into play, and the possible ways in which it would be justified are the following.

- There are many effective neighbours for each spin giving a large sum contributing to the local field. This will be true in a number of cases such as large spatial dimension (where one is of course closer to mean field) and long-range interactions for which even in dilute systems there are many effective neighbours. Therefore, one would expect that for certain short-range applications the theory may be expected to fail.

- When there are long correlations on the local field distribution one expects the central limit theorem, and thus the approximation, not to be applicable. Indeed for purely ferromagnetic interactions the correlation between local fields will become stronger as the temperature is reduced, we see that the mass in the two-point correlator does become rather small in three dimensions even if it does not become zero before the predicted spin-glass transition. In the non-ferromagnetic cases the correlations in the local fields do not become so strong and hence one may have more confidence in the method.

It is natural to enquire what we could say about the system beyond the level of the Gaussian variational analysis. Simulations of the basic Hamiltonian we start with are at an early stage [26], and the non-trivial dynamics usually associated with disordered systems has not yet been observed. The renormalization group could yield important information about the model. A naive application addresses the ferromagnetic transition because these are the obvious variables in the problem, but a perturbative approach leads to exactly the problem investigated by Dotsenko *et al* [9]. These authors found that the renormalization flow in a dilute ferromagnet was disrupted into a replica symmetry broken form. The precise interpretation of this observation is as yet unclear, but we hope that the analysis we present in this paper, based on the Gaussian variational method, will shed some light on the problem. One might hope that a more sophisticated approach to the renormalization, in which the appropriate variables for the spin-glass transition were isolated, would give new and interesting information. We postpone these issues to another publication [12].

We have observed continuous RSB in finite-dimensional models, and providing an interpretation by identifying the pure states in physical terms would be an interesting task. For this purpose, as mentioned earlier, it will be interesting to consider a potential that strictly vanishes beyond some radius, as the connection with percolation is clearer and one would expect to see a greater sensitivity on spatial dimension.

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Appendix A. The RS correlator for q_{ab}

In this appendix we resolve the equations (3.11):

$$\begin{aligned}\tilde{Q}_{abcd}(k) &= \rho\Omega_{abcd} + \frac{\rho}{2}\sum_{gh}\tilde{\Sigma}_{abgh}(k)\tilde{Q}_{ghcd}(k)\\ \tilde{\Sigma}_{abgh}(k) &= \sum_{ef}\Omega_{abef}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\tilde{G}_{eg}(p)\tilde{G}_{fh}(k-p)\end{aligned}\tag{A.1}$$

for the correlator $\tilde{Q}_{abcd}(k) = \langle\tilde{q}_{ab}(k)\tilde{q}_{cd}(-k)\rangle$ in the general paramagnetic RS case with $M = 0$, but arbitrary q.

Notice that although symmetry in $a \leftrightarrow b$ and $c \leftrightarrow d$ is manifest, the symmetry between pairs $(ab) \leftrightarrow (cd)$ is not. This could be rectified by an appropriate matrix multiplication, but it is simpler to leave.

These equations are a set of linear equations for the nine possible index combinations we have to consider since the diagonal terms are inevitably mixed into the equation (the missing symmetry would reduce this to the seven combinations, A, B, C, D, P, Q, R in the notation of [19]). The four-spin trace Ω_{abcd} is completely symmetric and takes values either 1 or q except for the case with all indices different where it is equal to $-1+2q+r$. $\Sigma_{abgh}(k)$ is defined by quadratics in g_0 and g_1. It is definitely not pair symmetric, but does satisfy $\Sigma_{(aa)(aa)} = \Sigma_{(aa)(bb)}$ and $\Sigma_{(aa)(ab)} = \Sigma_{(aa)(bc)}$. Other simple combinations are based on the longitudinal, $(P-4Q+3R)$, and replicon, $(P-2Q+R)$, mixtures of P: $\Sigma_{(ab)(ab)}$, Q: $\Sigma_{(ab)(ac)}$ and R: $\Sigma_{(ab)(bc)}$.

The 9×9 matrix equation splits into a 4×4 and a 5×5 block. The 4×4 part is simple giving, $\tilde{Q}_{(aa)(aa)} = \tilde{Q}_{(aa)(bb)} = \rho$. The 5×5 part contains the P, Q and R pieces and the longitudinal and replicon combinations of the Q's are particularly simple,

$$\begin{aligned}
\tilde{Q}_{lon} &= \frac{\rho(-2+2q+3r)}{\tilde{D}_{lon}} \\
\tilde{D}_{lon}(k) &= 1-\rho(-2+2q+3r)\int \frac{\mathrm{d}^d p}{(2\pi)^d}\tilde{g}_0(k-p)(\tilde{g}_0-2\tilde{g}_1)(p) \\
\tilde{Q}_{rep} &= \frac{\rho r}{\tilde{D}_{rep}} \\
\tilde{D}_{rep}(k) &= 1-\rho r\int \frac{\mathrm{d}^d p}{(2\pi)^d}\tilde{g}_0(k-p)\tilde{g}_0(p).
\end{aligned} \tag{A.2}$$

For the remaining index combinations we find that the pair symmetry is recovered and that $\tilde{Q}_{(aa)(ab)}(k)=\tilde{Q}_{(ab)(aa)}(k)=\tilde{Q}_{(aa)(bc)}(k)=\tilde{Q}_{(bc)(aa)}(k)$. The non-trivial momentum dependence of these correlators can be understood since the q_{aa} operator just checks for the presence of a spin, whereas the q_{ab} is the squared magnetization and obviously depends on the presence of a nearby spin.

The full solution is,

$$\begin{aligned}
\tilde{Q}_A &= \tilde{Q}_{(aa)(aa)} = \tilde{Q}_B = \tilde{Q}_{(aa)(bb)} = \rho \\
\tilde{Q}_C &= \tilde{Q}_{(aa)(ab)} = \tilde{Q}_D = \tilde{Q}_{(aa)(bc)} = \rho q + 2\tilde{\Sigma}_C\tilde{Q}_{lon} \\
\tilde{Q}_P &= \tilde{Q}_{(ab)(ab)} = \frac{1}{\tilde{D}_{lon}}(\rho + 2(\tilde{\Sigma}_{lon}-\tilde{\Sigma}_{rep})(\tilde{Q}_{lon}-3\tilde{Q}_{rep}) + 2\tilde{\Sigma}_{CD}\tilde{Q}_C \\
&\quad + 2\tilde{\Sigma}_R\tilde{Q}_{lon}) \\
\tilde{Q}_Q &= \tilde{Q}_{(ab)(ac)} = \frac{1}{\tilde{D}_{lon}}(\rho + (\tilde{\Sigma}_{lon}-2\tilde{\Sigma}_{rep}+\tfrac{1}{2})(\tilde{Q}_{lon}-3\tilde{Q}_{rep}) + 2\tilde{\Sigma}_{CD}\tilde{Q}_C \\
&\quad + 2\tilde{\Sigma}_R\tilde{Q}_{lon}) \\
\tilde{Q}_R &= \tilde{Q}_{(ab)(ac)} = \tilde{Q}_{rep} + \frac{1}{\tilde{D}_{lon}}(\rho + (-2\tilde{\Sigma}_{rep}+1)(\tilde{Q}_{lon}-3\tilde{Q}_{rep}) + 2\tilde{\Sigma}_{CD}\tilde{Q}_C \\
&\quad + 2\tilde{\Sigma}_R\tilde{Q}_{lon}).
\end{aligned} \tag{A.3}$$

Where besides the longitudinal and replicon components some of the other Σ combinations are needed,

$$\begin{aligned}
\tilde{\Sigma}_C(k) &= \rho\int \frac{\mathrm{d}^d p}{(2\pi)^d}\tilde{g}_0(k-p)\tilde{g}_1(p) + \tfrac{1}{2}\rho q\int \frac{\mathrm{d}^d p}{(2\pi)^d}\tilde{g}_0(k-p)(\tilde{g}_0-2\tilde{g}_1)(p) \\
\tilde{\Sigma}_{rep}(k) &= \tfrac{1}{2}(1-\tilde{D}_{rep}) \\
\tilde{\Sigma}_{lon}(k) &= \tfrac{1}{2}(1-\tilde{D}_{lon}) \\
\tilde{\Sigma}_{CD}(k) &= \rho(-2+2q+3r)\int \frac{\mathrm{d}^d p}{(2\pi)^d}\tilde{g}_0(k-p)\tilde{g}_1(p) \\
\tilde{\Sigma}_R(k) &= \rho(2+2q+3r)\int \frac{\mathrm{d}^d p}{(2\pi)^d}\tilde{g}_0(k-p)\tilde{g}_0(p) \\
&\quad + 3\rho(1-q-r)\int \frac{\mathrm{d}^d p}{(2\pi)^d}\tilde{g}_0(k-p)\tilde{g}_1(p) \\
&\quad + \tfrac{1}{2}\rho(-1+2q+r)\int \frac{\mathrm{d}^d p}{(2\pi)^d}\tilde{g}_1(k-p)\tilde{g}_1(p).
\end{aligned} \tag{A.4}$$

These final forms for each individual component are not especially illuminating, but it is worth nothing that they have denominators only of the form D_{lon}, D_{rep}, D_{lon}^2 and $D_{rep}D_{lon}$.

The replicon denominator vanishes first,

$$\tilde{D}_{rep}(k=0) = 1 - \rho r \int \frac{\mathrm{d}^d p}{(2\pi)^d} \tilde{g}_0^2(p) = 1 - \frac{r g_1}{q}. \tag{A.5}$$

So we recover the AT condition as the divergence of this correlator.

The case in which q vanishes is particularly simple, $\tilde{D}_{rep} = \tilde{D}_{lon} = \tilde{D}$ and only the P combination (and $\tilde{Q}_{(aa)(bb)} = \rho$) remains non-vanishing,

$$\tilde{Q}_{(ab)(ab)}(k) = \frac{\rho}{\tilde{D}}. \tag{A.6}$$

Similar techniques could be used to analyse the correlator in the RS but ferromagnetic phase and also for solutions with one step of RSB.

Appendix B. Stability of RS solutions

We present the usual stability analysis based on the eigenvalues for small fluctuations about a RS solution along the lines of the work of de Almeida and Thouless [19]. We therefore need the Hessian matrix $H_{(ab)(cd)}$:

$$\frac{\delta^2 F_{var}}{\delta G_{ab}(k)\delta G_{cd}(k')} \propto (G_{ac}^{-1}(k)G_{db}^{-1}(k) + G_{ad}^{-1}(k)G_{bc}^{-1}(k))\delta(k-k') - \frac{\rho}{(2\pi)^d}\Omega_{abcd}. \tag{B.1}$$

For RS solutions the components (in the notation of [19] where all indices are different) of the Hessian matrix are given by:

$$\begin{aligned}
A &= H_{(aa)(aa)} = 2\left(\frac{1}{\tilde{g}_0} - \frac{\tilde{g}_1}{\tilde{g}_0^2}\right)^2 \delta(k-k') - \frac{\rho}{(2\pi)^d}\Omega \\
B &= H_{(aa)(bb)} = 2\left(\frac{\tilde{g}_1}{\tilde{g}_0^2}\right)^2 \delta(k-k') - \frac{\rho}{(2\pi)^d}\Omega \\
C &= H_{(aa)(ab)} = H_{(ab)(aa)} = -2\left(\frac{\tilde{g}_1}{\tilde{g}_0^2}\right)\left(\frac{1}{\tilde{g}_0} - \frac{\tilde{g}_1}{\tilde{g}_0^2}\right)\delta(k-k') - \frac{\rho}{(2\pi)^d}\Omega_{ab} \\
D &= H_{(aa)(bc)} = H_{(bc)(aa)} = 2\left(\frac{\tilde{g}_1}{\tilde{g}_0^2}\right)^2 \delta(k-k') - \frac{\rho}{(2\pi)^d}\Omega_{ab} \\
P &= H_{(ab)(ab)} = \left(\left(\frac{\tilde{g}_1}{\tilde{g}_0^2}\right)^2 + \left(\frac{1}{\tilde{g}_0} - \frac{\tilde{g}_1}{\tilde{d}_0^2}\right)^2\right)\delta(k-k') - \frac{\rho}{(2\pi)^d}\Omega \\
Q &= H_{(ab)(ac)} = -\left(\frac{\tilde{g}_1}{\tilde{g}_0^2}\right)\left(\frac{1}{\tilde{g}_0} - \frac{2\tilde{g}_1}{\tilde{g}_0^2}\right)\delta(k-k') - \frac{\rho}{(2\pi)^d}\Omega_{bc} \\
R &= H_{(ab)(cd)} = 2\left(\frac{\tilde{g}_1}{\tilde{g}_0^2}\right)^2 \delta(k-k') - \frac{\rho}{(2\pi)^d}\Omega_{abcd}.
\end{aligned} \tag{B.2}$$

Diagonalization in replica space gives the replicon operator as the combination $P - 2Q + R$. It is convenient to introduce the positive weight function $\tilde{g}_0^{-2}$ into the k-space eigenequation to obtain,

$$\frac{1}{\tilde{g}_0^2(k)} f(k) - \frac{\rho r}{(2\pi)^d} \int \mathrm{d}^d k' f(k') = \lambda_{rep} \frac{1}{\tilde{g}_0^2(k)} f(k). \tag{B.3}$$

Where $r = (1 - 2\Omega_{ab} + \Omega_{abcd})$ is defined in (4.10). In the case where $\int f = 0$, the eigenvalue is clearly positive, otherwise we integrate over k to obtain the condition,

$$\rho r \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{g}_0^2 = \frac{r}{q} g_1 < 1. \tag{B.4}$$

The longitudinal mode is slightly more complicated because the diagonal terms A, B, C, D contribute. The equations for the eigenvalues are,

$$\begin{aligned} &(A-B)f_0 + [(A-B) - 2(C-D)]f_1 = \lambda_{lon}(f_0 + f_1) \\ &2(C-D)f_0 + 2[(C-D) + (P - 4Q + 3R)]f_1 = \lambda_{lon} f_1. \end{aligned} \tag{B.5}$$

With some rearrangement, and the same weight function, we find,

$$\begin{aligned} &\frac{(\tilde{g}_0 - 2\tilde{g}_1)}{\tilde{g}_0} f_0 + f_1 = \lambda_{lon}(f_0 + f_1) \\ &f_0 + \rho(-2 + 2q + 3r)\tilde{g}_0^2 \int f_1(k') = \lambda_{lon} f_0. \end{aligned} \tag{B.6}$$

An inspection of these equations in the same way as for the replicon leads to the following condition for stability,

$$\rho(-2 + 2q + 3r) \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{g}_0(\tilde{g}_0 - 2\tilde{g}_1) < 1. \tag{B.7}$$

For the diagonal RS solution in which q vanishes, $r = 1$, we find that both conditions (B.4) and (B.7) are the same and we recover the same line (4.7) for the spin-glass transition.

References

[1] Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1972; 1977 *Phys. Rev.* B **17** 4384
[2] Edwards S F and Anderson P W 1975 *J. Physique* F **5** 965
[3] Nieuwenhuizen Th M 1993 *Europhys. Lett.* **24** 797
[4] Dean D S and Lancaster D J 1996 *Phys. Rev. Lett.* **77** 3037
[5] Dean D S and Lancaster D J in preparation
[6] Mézard M, Parisi G and Virasoro M A, 1987 *Spin Glass Theory and Beyond (Lecture Notes in Physics 9)* (Singapore: World Scientific)
[7] Fisher D S and Huse D A 1986 *Phys. Rev. Lett.* **56** 1601; 1988 *Phys. Rev.* B **38** 373
[8] Mézard M and Young A P 1992 *Europhys. Lett.* **18** 653
Mézard M and Monasson R 1994 *Phys. Rev.* B **50** 7199
[9] Dotsenko V, Harris A B, Sherrington D and Stinchcombe R B 1995 *J. Phys. A: Math. Gen.* **28** 3093
[10] Le Dousal P and Giamarchi T 1995 *Phys. Rev. Lett.* **74** 606
Kierfeld J 1995 *J. Physique* I **5** 379
[11] Fischer K H and Hertz J A 1991 *Spin Glasses* (Cambridge: Cambridge University Press) ch 11
[12] Dean D S and Lancaster D J in preparation
[13] Pytte E and Rudnick J 1979 *Phys. Rev.* B **19** 3603
[14] Feynman R P 1987 *Variational Calculations in Quantum Field Theory (Proceedings)* (Wangerooge: World Scientific)
[15] Mézard M and Parisi G 1991 *J. Physique* I **1** 809
[16] Shakhnovich E I and Gutin A M 1989 *J. Phys. A: Math. Gen.* **22** 1647
[17] Parisi G 1988 *Statistical Field Theory* (Reading, MA: Addison-Wesley)
[18] Kondor I, De Dominicis C and Temesvári T 1994 *J. Phys. A: Math. Gen.* **27** 7569
[19] de Almeida J R L and Thouless D J 1978 *J. Phys. A: Math. Gen.* **11** 983
[20] Sherrington D 1980 *Phys. Rev.* B **22** 5553
[21] Parisi G 1980 *J. Phys. A: Math. Gen.* **13** 1101
[22] Thouless D J, de Almeida J R L and Kosterlitz J M 1980 *J. Phys. C: Solid State Phys.* **13** 3271
[23] Parisi G 1980 *J. Phys. A: Math. Gen.* **13** L115
Duplantier B 1981 *J. Phys. A: Math. Gen.* **14** 283
[24] Viana L and Bray A J 1985 *J. Phys. C: Solid State Phys.* **18** 3037
De Dominicis C and Mottishaw P 1987 *J. Phys. A: Math. Gen.* **20** L1267
[25] Monasson R 1995 *Phys. Rev. Lett.* **75** 2847
[26] Iori G Private communication
[27] McLenaghan I R and Sherrington D 1984 *J. Phys. C: Solid State Phys.* **17** 1531